

TOPOLOGICAL FORMULA OF THE LOOP EXPANSION OF THE COLORED JONES POLYNOMIALS

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ABSTRACT. We give a topological formula of the loop expansion of the colored Jones polynomials by using identification of generic quantum \mathfrak{sl}_2 representation with homological representations. This gives a direct topological proof of the Melvin-Morton-Rozansky conjecture, and a connection between entropy of braids and quantum representations.

1. INTRODUCTION

For $\alpha \in \{2, 3, 4, \dots\}$ and an oriented knot K in S^3 , let $J_{K,\alpha}(q) \in \mathbb{Z}[q, q^{-1}]$ be the α -colored Jones polynomial of K normalized so that $J_{\text{Unknot},\alpha}(q) = 1$. As Melvin-Morton proved [MeMo], by putting $q = e^{\hbar}$, the colored Jones polynomials can be expanded as a power series of two independent variables $\hbar\alpha$ and \hbar , as

$$J_{K,\alpha}(e^{\hbar}) = \sum_{i=0}^{\infty} D^{(i)}(\hbar\alpha) \hbar^i = \sum_{i=0}^{\infty} \left(\sum_{k=0}^{\infty} d_k^{(i)}(\hbar\alpha)^k \right) \hbar^i.$$

Further, we put $z = e^{\hbar\alpha}$ and write the colored Jones polynomials as a function on \hbar and z ,

$$CJ_K(z, \hbar) = J_{K,\alpha}(e^{\hbar}) = \sum_{i=0}^{\infty} V_K^{(i)}(z) \hbar^i.$$

We call CJ_K the *colored Jones function* or, the *loop expansion of quantum \mathfrak{sl}_2 -invariant* since it coincides with the \mathfrak{sl}_2 weight system reduction of the loop expansion of the Kontsevich invariant. In particular, the i -th coefficient $V^{(i)}(z)$ corresponds to the $(i+1)$ -st loop part of the loop expansion of the Kontsevich invariant [Oh1].

Let $\nabla_K(z)$ be the Conway polynomial of K , characterized by the skein relation

$$\nabla_{\text{crossing}}(z) - \nabla_{\text{crossing}}(z) = z \nabla_{\text{crossing}}(z), \quad \nabla_{\text{Unknot}}(z) = 1$$

and let $\Delta_K(z) = \nabla_K(z^{\frac{1}{2}} - z^{-\frac{1}{2}})$ be the Alexander-Conway polynomial. The Alexander-Conway polynomial appears as one of the basic building block of CJ_K . The Melvin-Morton-Rozansky conjecture [MeMo] (MMR conjecture, in short), proven in [BG], states that the $V^{(0)}(z)$ is equal to $\Delta_K(z)^{-1}$. More generally, $V^{(i)}(z)$ is a rational function whose denominator is $\Delta_K(z)^{2i+1}$ [Ro1].

In a theory of quantum invariants, the appearance of the Alexander-Conway polynomial is well-understood. The aforementioned rationality of $V^{(i)}(z)$ follows from Rozansky's rationality conjecture [Ro2] of the loop expansion of Kontsevich invariant, proven in [Kri]: The Aarhus integral computation of Kontsevich (or, LMO) invariant, based on a surgery presentation of knots, provides the desired rationality (see [GK, Section 1.2] for a brief summary of Kricker's argument).

The clasper surgery [Ha] explains a geometric connection between the loop expansion and infinite cyclic covering [GR]. A null-clasper, a clasper with null-homologous leaves in the knot complement, lifts to a clasper in the infinite cyclic covering, and the loop expansion nicely behaves under the clasper surgery along null-claspers. Thus schematically speaking, the loop expansion provides a \mathbb{Z} -equivariant quantum invariants [GR] (for example, the 2-loop part can be interpreted as the \mathbb{Z} -equivariant Casson invariant, as discussed in [Oh2]), so it is not surprising that the Alexander-Conway polynomial appears in the loop expansion.

2010 *Mathematics Subject Classification.* Primary 57M27, Secondary 37B40, 20F36, 81R50.

Key words and phrases. Colored Jones polynomial, Loop expansion, homological representation of the braid groups, entropy.

Nevertheless, it is still mysterious why the Alexander polynomial appears in such a particular and direct form. Even for the MMR conjecture, the simplest and the most fundamental rationality result, the situation is not so good as we want. In a known proof, one uses quantum-invariant-like treatment of the Alexander-Conway polynomial such as, state-sum, R-matrix, or weight systems so its topological content is often indirect.

In this paper, we give a topological formula of CJ_K by using homological braid group representations (Theorem 3.1). Our starting point is a recent result in [I, Koh] that identifies certain homological representations introduced by Lawrence [La] with generic $U_q(\mathfrak{sl}_2)$ representations. This allows us to translate a construction of the colored Jones function in terms of corresponding homological representations. Also, in Section 4 we discuss a connection among the entropy of braids, quantum representations, and the growth of quantum \mathfrak{sl}_2 invariants inspired from topological point of view.

One may notice that our approach is similar to Lawrence-Bigelow's approach of Jones polynomial [Big2, La2], but there are several critical differences: We give a formula of the loop expansion CJ_K but do not provide a formula of each individual colored Jones polynomials. Our formula uses closed braid representatives, whereas Lawrence-Bigelow description uses plat representatives and intersection products.

It should be emphasized that quantum representations coming from finite dimensional $U_q(\mathfrak{sl}_2)$ -module is *not* identified with homological representation. This is the reason why we do not have a direct topological formula of usual colored Jones polynomials.

Our topological description leads to several insights. First, the MMR conjecture is now obtained as a direct consequence of our topological formula. By putting $\hbar = 0$, topological considerations show that the homological representations is equal to the symmetric powers of the reduced Burau representation, so they naturally lead to the Alexander-Conway polynomial. Second, our formula gives a new and direct way to calculate $CJ_K(z, \hbar)$ without knowing or computing individual colored Jones polynomial $J_{K,\alpha}(q)$ or appealing surgery presentation of knots, although a general calculation is still difficult.

ACKNOWLEDGMENTS

The author was partially supported by JSPS Grant-in-Aid for Research Activity start-up, Grant Number 25887030. He would like to thank Tomotada Ohtsuki, Jun Murakami and Hitoshi Murakami for stimulating discussion and comments.

2. A TOPOLOGICAL DESCRIPTION OF GENERIC QUANTUM \mathfrak{sl}_2 REPRESENTATION

In this section we review the result in [I] that identifies a generic quantum \mathfrak{sl}_2 representation given in [JK] with Lawrence's homological representation and some additional arguments to treat non-generic case.

Throughout the paper, we use the following notations and conventions. The q -numbers, q -factorials, and q -binomial coefficients are defined by

$$[n]_q = \frac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}, \quad [n]_q! = [n]_q [n-1]_q \cdots [2]_q [1]_q, \quad \begin{bmatrix} n \\ j \end{bmatrix}_q = \frac{[n]_q!}{[n-j]_q! [j]_q!},$$

respectively. This convention is different from one in [I, JK]. The quantum parameter q in this paper corresponds to q^2 in [I, JK]. We always assume that the braid group B_n is acting from left.

Let R be a commutative ring. For R -modules (resp. RB_n -modules) V and W , we denote $V \cong_{\mathcal{Q}} W$ if they are isomorphic over the quotient field \mathcal{Q} of R . Namely, $C \cong_{\mathcal{Q}} W$ implies $V \otimes_R \mathcal{Q}$ and $W \otimes_R \mathcal{Q}$ are isomorphic as \mathcal{Q} -modules (resp. $\mathcal{Q}B_n$ -module).

For a subring $R \subset \mathbb{C}$, let $R[x^{\pm 1}]$ be the Laurent polynomial ring, and for an $R[x^{\pm 1}]$ -module V and $c \in \mathbb{C}$, we denote the specialization of the variable x to complex parameter c by $V|_{x=c}$.

2.1. Generic quantum representation. Let $\mathbb{C}[[\hbar]]$ be the algebra of the complex formal power series in one variable \hbar , and we put $q = e^{\hbar}$, as usual. A quantum enveloping algebra $U_{\hbar}(\mathfrak{sl}_2)$ is a

topological Hopf algebra over $\mathbb{C}[[\hbar]]$ generated by H, E, F subjected to the relations

$$\begin{cases} [H, E] = 2E, & [H, F] = -2F, \\ [E, F] = \frac{\sinh(\frac{\hbar H}{2})}{\sinh(\frac{\hbar}{2})} = \frac{e^{\frac{\hbar H}{2}} - e^{-\frac{\hbar H}{2}}}{e^{\frac{\hbar}{2}} - e^{-\frac{\hbar}{2}}} \end{cases}$$

$U_{\hbar}(\mathfrak{sl}_2)$ is a quasi-triangular topological Hopf algebra and a *universal R -matrix* $\mathcal{R} \in U_{\hbar}(\mathfrak{sl}_2) \otimes U_{\hbar}(\mathfrak{sl}_2)$ is given by

$$(2.1) \quad \mathcal{R} = e^{\frac{\hbar}{4}(H \otimes H)} \left(\sum_{n=0}^{\infty} q^{\frac{n(n-1)}{4}} \frac{(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^n}{[n]_q!} E^n \otimes F^n \right).$$

(Strictly speaking, here we need to use the topological tensor product $\widetilde{\otimes}$, the \hbar -adic completion of $U_{\hbar}(\mathfrak{sl}_2) \otimes U_{\hbar}(\mathfrak{sl}_2)$. To make notation simple, in the rest of the paper \otimes should be regarded as the topological tensor product, if we should do so.)

For $\lambda \in \mathbb{C}^*$, let V_{λ} be the Verma module of highest weight λ , a topologically free $U_{\hbar}(\mathfrak{sl}_2)$ -module generated by a highest weight vector v_0 with $Hv_0 = \lambda v_0$ and $Ev_0 = 0$.

Now let us regard λ as an abstract variable. Let $\widehat{V_{\hbar, \lambda}}$ be a $\mathbb{C}[\lambda][[\hbar]]$ -module freely generated by $\{\widehat{v}_0, \widehat{v}_1, \dots\}$, equipped with an $U_{\hbar}(\mathfrak{sl}_2)$ -module structure

$$(2.2) \quad \begin{cases} H\widehat{v}_i = (\lambda - 2i)\widehat{v}_i \\ E\widehat{v}_i = \widehat{v}_{i-1} \\ F\widehat{v}_i = [i+1]_q[\lambda - i]_q\widehat{v}_{i+1}. \end{cases}$$

Here we put

$$[\lambda - i]_q = \frac{\sinh(\frac{1}{2}\hbar(\lambda - i))}{\sinh(\frac{1}{2}\hbar)} = \frac{e^{\frac{1}{2}\hbar(\lambda - i)} - e^{-\frac{1}{2}\hbar(\lambda - i)}}{e^{\frac{1}{2}\hbar} - e^{-\frac{1}{2}\hbar}}.$$

We call $\widehat{V_{\hbar, \lambda}}$ a *generic Verma module*.

For $j = 0, 1, \dots$, define

$$v_j = [\lambda]_q[\lambda - 1]_q \cdots [\lambda - j + 1]_q \widehat{v}_j$$

and let $V_{\hbar, \lambda}$ be the sub $U_{\hbar}(\mathfrak{sl}_2)$ -module of $\widehat{V_{\hbar, \lambda}}$ spanned by $\{v_0, v_1, \dots\}$, with the action of $U_{\hbar}(\mathfrak{sl}_2)$ given by

$$(2.3) \quad \begin{cases} Hv_i = (\lambda - 2i)v_i \\ Ev_i = [\lambda + 1 - i]_q v_{i-1} \\ Fv_i = [i+1]_q v_{i+1}. \end{cases}$$

For $c \notin \mathbb{C}^* - \{1, 2, \dots\}$, $\widehat{V_{\hbar, \lambda}}|_{\lambda=c}$ is isomorphic to $V_{\hbar, \lambda}|_{\lambda=c}$ because $[\lambda]_q[\lambda - 1]_q \cdots [\lambda - j + 1]_q$ is invertible for all j . On the other hand, for $c \in \{1, 2, \dots\}$, $v_j = 0$ if $j > c$ and (2.3) shows that $V_{\hbar, \lambda}|_{\lambda=c}$ is nothing but the standard irreducible $U_{\hbar}(\mathfrak{sl}_2)$ -module of dimension $(c+1)$ whereas $\widehat{V_{\hbar, \lambda}}|_{\lambda=c}$ is infinite dimensional representation.

Let us define $R : \widehat{V_{\hbar, \lambda}} \otimes \widehat{V_{\hbar, \lambda}} \rightarrow \widehat{V_{\hbar, \lambda}} \otimes \widehat{V_{\hbar, \lambda}}$ by $R = e^{-\frac{1}{4}\hbar\lambda^2} T\mathcal{R}$, where $T : \widehat{V_{\hbar, \lambda}} \otimes \widehat{V_{\hbar, \lambda}} \rightarrow \widehat{V_{\hbar, \lambda}} \otimes \widehat{V_{\hbar, \lambda}}$ is the transposition map $T(v \otimes w) = w \otimes v$, and \mathcal{R} is the universal R -matrix (2.1).

By putting $z = q^{\lambda-1} = e^{\hbar(\lambda-1)}$, the action of R is given by the formula

$$(2.4) \quad \begin{cases} R(\widehat{v}_i \otimes \widehat{v}_j) = z^{-\frac{i+j}{2}} q^{-\frac{i+j}{2}} \sum_{n=0}^i q^{(i-n)(j+n)} q^{\frac{n(n-1)}{4}} \frac{[n+j]_q!}{[n]_q! [j]_q!} \prod_{k=0}^{n-1} (z^{\frac{1}{2}} q^{-\frac{1+k+j}{2}} - z^{-\frac{1}{2}} q^{\frac{1+k+j}{2}}) \widehat{v}_{j+n} \otimes \widehat{v}_{i-n}. \\ R(v_i \otimes v_j) = z^{-\frac{i+j}{2}} q^{-\frac{i+j}{2}} \sum_{n=0}^i q^{(i-n)(j+n)} q^{\frac{n(n-1)}{4}} \frac{[n+j]_q!}{[n]_q! [j]_q!} \prod_{k=0}^{n-1} (z^{\frac{1}{2}} q^{\frac{1-i+k}{2}} - z^{-\frac{1}{2}} q^{-\frac{1-i+k}{2}}) v_{j+n} \otimes v_{i-n} \end{cases}$$

Let $\mathbb{L} = \mathbb{Z}[q^{\pm 1}, z^{\pm 1}] = \mathbb{Z}[e^{\pm \hbar}, e^{\pm \hbar(\lambda-1)}] \subset \mathbb{C}[\lambda][[\hbar]]$ and let $V_{\mathbb{L}}$ and $\widehat{V_{\mathbb{L}}}$ be the sub free \mathbb{L} -module of $\widehat{V_{\hbar, \lambda}}$ and $V_{\hbar, \lambda}$, spanned by $\{\widehat{v}_0, \dots\}$ and $\{v_0, \dots\}$, respectively.

Since all the coefficients of the action of R (2.4) lie in \mathbb{L} , $\widehat{V_{\mathbb{L}}}$ and $V_{\mathbb{L}}$ are equipped with an $\mathbb{L}B_n$ -module structure. We denote the corresponding braid group representations by

$$\widehat{\varphi}_{\mathbb{L}} : B_n \rightarrow \mathrm{GL}(\widehat{V_{\mathbb{L}}}^{\otimes n}), \quad \varphi_{\mathbb{L}} : B_n \rightarrow \mathrm{GL}(V_{\mathbb{L}}^{\otimes n}).$$

These are decomposed as finite dimensional representations as follows. For $m \geq 0$, define $\widehat{V}_{n,m} \subset \widehat{V}_{\mathbb{L}}^{\otimes n}$ and $V_{n,m} \subset V_{\mathbb{L}}^{\otimes n}$ by

$$\begin{cases} \widehat{V}_{n,m} = \ker(q^H - q^{\frac{n\lambda-2m}{2}}) = \text{span}\{\widehat{v}_{i_1} \otimes \cdots \otimes \widehat{v}_{i_n} \mid i_1 + \cdots + i_n = m\}. \\ V_{n,m} = \ker(q^H - q^{\frac{n\lambda-2m}{2}}) = \text{span}\{v_{i_1} \otimes \cdots \otimes v_{i_n} \mid i_1 + \cdots + i_n = m\} \end{cases}$$

By (2.4), the B_n -action preserves both $\widehat{V}_{n,m}$ and $V_{n,m}$ so we have linear representations

$$\varphi_{n,m}^V : B_n \rightarrow \text{GL}(\widehat{V}_{n,m}) \quad \text{and} \quad \varphi_{n,m}^V : B_n \rightarrow \text{GL}(V_{n,m}).$$

We call the $\mathbb{L}B_n$ -module $\widehat{V}_{n,m}$ the (*generic*) *weight space* of weight $q^{\frac{n\lambda-2m}{2}}$.

By definition, as $\mathbb{L}B_n$ -modules, $\widehat{V}_{\mathbb{L}}^{\otimes n}$ and $V_{\mathbb{L}}^{\otimes n}$ split as

$$(2.5) \quad \begin{cases} \widehat{V}_{\mathbb{L}}^{\otimes n} \cong \bigoplus_{m=0}^{\infty} \widehat{V}_{n,m} \\ V_{\mathbb{L}}^{\otimes n} \cong \bigoplus_{m=0}^{\infty} V_{n,m} \end{cases}$$

Finally, we define the *space of (generic) null vectors* $\widehat{W}_{n,m}$ by

$$\widehat{W}_{n,m} = \text{Ker}(E) \cap \widehat{V}_{n,m}.$$

Since the action of B_n commutes with the action of $U_q(\mathfrak{sl}_2)$, we have linear representation

$$\varphi_{n,m}^W : B_n \rightarrow \text{GL}(\widehat{W}_{n,m}).$$

In [JK, Lemma 13], it is shown that for $k = 1, \dots, m$, the map $F^{m-k} : \widehat{W}_{n,k} \rightarrow \widehat{V}_{n,m}$ is injection and that over the quotient field, $\widehat{V}_{n,m}$ splits as

$$(2.6) \quad \widehat{V}_{n,m} \cong_{\mathbb{Q}} \bigoplus_{k=0}^m F^{m-k} \widehat{W}_{n,k} \cong_{\mathbb{Q}} \bigoplus_{k=0}^m \widehat{W}_{n,k},$$

hence combining with (2.5), we conclude that the $\mathbb{L}B_n$ -module $\widehat{V}_{\mathbb{L}}^{\otimes n}$ splits, over the quotient field,

$$(2.7) \quad \widehat{V}_{\mathbb{L}}^{\otimes n} \cong_{\mathbb{Q}} \bigoplus_{m=0}^{\infty} \bigoplus_{k=0}^m \widehat{W}_{n,k}.$$

2.2. Lawrence's homological representations. Here we briefly review the definition of (geometric) Lawrence's representation $L_{n,m}$. An explicit matrix of $L_{n,m}(\sigma_i)$ and some details will be given in Appendix.

For $i = 1, 2, \dots, n$, let $p_i = i \in \mathbb{C}$ and $D_n = \{z \in \mathbb{C} \mid |z| \leq n+1\} - \{p_1, \dots, p_n\}$ be the n -punctured disc. We identify the braid group B_n with the mapping class group of D_n so that the standard generator σ_i corresponds to the *right-handed* half Dehn twist that interchanges the i -th and $(i+1)$ -st punctures.

For $m > 0$, let $C_{n,m}$ be the unordered configuration space of m -points in D_n ,

$$C_{n,m} = \{(z_1, \dots, z_m) \in D_n \mid z_i \neq z_j \ (i \neq j)\} / S_m$$

where S_m is the symmetric group acting as permutations of the indices. For $i = 1, \dots, n$, let $d_i = (n+1)e^{(\frac{3}{2}+i\varepsilon)\pi\sqrt{-1}} \in \partial D_n$ where $\varepsilon > 0$ is sufficiently small number, and we take $\mathbf{d} = \{d_1, \dots, d_m\}$ as a base point of $C_{n,m}$.

The first homology group $H_1(C_{n,m}; \mathbb{Z})$ is isomorphic to $\mathbb{Z}^{\oplus n} \oplus \mathbb{Z}$, where the first n components correspond to the meridians of the hyperplanes $\{z_1 = p_i\}$ ($i = 1, \dots, n$) and the last component corresponds to the meridian of the discriminant $\bigcup_{1 \leq i < j \leq n} \{z_i = z_j\}$.

Let $\alpha : \pi_1(C_{n,m}) \rightarrow \mathbb{Z}^2 = \langle x, d \rangle$ be the homomorphism obtained by composing the Hurewicz homomorphism $\pi_1(C_{n,m}) \rightarrow H_1(C_{n,m}; \mathbb{Z})$ and the projection

$$H_1(C_{n,m}; \mathbb{Z}) = \mathbb{Z}^{\oplus n} \oplus \mathbb{Z} = \langle x_1, \dots, x_n \rangle \oplus \langle d \rangle \rightarrow \langle x_1 + \cdots + x_n \rangle \oplus \langle d \rangle = \langle x \rangle \oplus \langle d \rangle = \mathbb{Z} \oplus \mathbb{Z}.$$

Let $\pi : \widetilde{C}_{n,m} \rightarrow C_{n,m}$ be the covering corresponding to $\text{Ker } \alpha$. We fix a lift $\widetilde{\mathbf{d}} \in \pi^{-1}(\mathbf{d}) \subset \widetilde{C}_{n,m}$ and use $\widetilde{\mathbf{d}}$ as a base point of $\widetilde{C}_{n,m}$. By identifying x and d as deck translations, $H_m(\widetilde{C}_{n,m}; \mathbb{Z})$ has

a structure of $\mathbb{Z}[x^{\pm 1}, d^{\pm 1}]$ -module. Actually, it is known that $H_m(\tilde{C}_{n,m}; \mathbb{Z})$ is a free $\mathbb{Z}[x^{\pm 1}, d^{\pm 1}]$ -module of rank $\binom{m+n-2}{m}$.

We will actually use $H_m^{lf}(\tilde{C}_{n,m}; \mathbb{Z})$, the homology of locally finite chains, and consider a free sub $\mathbb{Z}[x^{\pm 1}, d^{\pm 1}]$ -module $\mathcal{H}_{n,m} \subset H_m^{lf}(\tilde{C}_{n,m}; \mathbb{Z})$ of rank $\binom{m+n-2}{m}$, spanned by homology classes represented by certain geometric objects called *mutliforks*. The subspace $\mathcal{H}_{n,m}$ is preserved by B_n actions, hence by using a natural basis of $\mathcal{H}_{n,m}$ called *standard mutliforks*, we get a linear representation

$$L_{n,m}: B_n \rightarrow \mathrm{GL}(\mathcal{H}_{n,m}) = \mathrm{GL}(\binom{m+n-2}{m}; \mathbb{Z}[x^{\pm 1}, d^{\pm 1}]).$$

which we call (*Geometric*) *Lawrence's representation*.

In the case $m = 1$ the discriminant $\bigcup_{1 \leq i < j \leq n} \{z_i = z_j\}$ is empty so the variable d does not appear. The representation

$$L_{n,1}: B_n \rightarrow \mathrm{GL}(n-1; \mathbb{Z}[x^{\pm 1}]).$$

coincides with the reduced Burau representation. The representation $L_{n,2}$ is often called the *Lawrence-Krammer-Bigelow representation*, which is extensively studied in [Big1, Kra, Kra2] and known to be faithful.

Remark 2.1. In general, the braid group representations $\mathcal{H}_{n,m}$, $H_m(\tilde{C}_{n,m}; \mathbb{Z})$ and $H_m^{lf}(\tilde{C}_{n,m}; \mathbb{Z})$ are not isomorphic each other. However, there is an open dense subset $U \subset \mathbb{C}^2$ such that if we specialize x and d to complex parameters in U , then these three representations are isomorphic [Koh]. Namely, all representations are *generically* identical. In particular, they are all isomorphic over the quotient field, $\mathcal{H}_{n,m} \cong_{\mathbb{Q}} H_m(\tilde{C}_{n,m}; \mathbb{Z}) \cong_{\mathbb{Q}} H_m^{lf}(\tilde{C}_{n,m}; \mathbb{Z})$.

The following well-known result will explain why the MMR conjecture is true.

Proposition 2.2. *When we specialize $d = -1$, then the Lawrence's representation $L_{n,m}$ is equal to $\mathrm{Sym}^m L_{n,1}$, the m -th symmetric power of the reduced Burau representation $L_{n,1}$.*

Proposition 2.2 is directly seen by the explicit matrix formula of $L_{n,m}$ (A.1) in Appendix.

Roughly saying, when we specialized $d = -1$, that is, when we ignore the effect of discriminant $\bigcup_{1 \leq i < j \leq n} \{z_i = z_j\}$, we forget an interaction of points. Then a natural inclusion $C_{n,m} \rightarrow C_{n,1}^m / S_m$ induces an isomorphism

$$H_m(\widetilde{C_{n,m}}; \mathbb{Z})|_{d=-1} \rightarrow H_m(\widetilde{C_{n,1}^m} / S_m; \mathbb{Z}) \cong H_1(\widetilde{C_{n,1}}; \mathbb{Z})^{\otimes m} / S_m = \mathrm{Sym}^m H_1(\widetilde{C_{n,1}}; \mathbb{Z})$$

of the braid group representations.

Here, we remark that somewhat confusing minus sign of d comes from the convention of the orientation of submanifold representing an element of $H_m(\widetilde{C_{n,m}}; \mathbb{Z})$, as we will explain in Appendix.

2.3. Identification and specializations of quantum and homological representations.

Here we summarize relations of braid group representations introduced in previous sections. First, generically quantum representation is identified with Lawrence's representation.

Theorem 2.3. [I, Corollary 4.6], [Koh] *As a braid group representation, there is an isomorphism*

$$\widehat{W_{n,m}} \cong_{\mathbb{Q}} \mathcal{H}_{n,m}|_{x=z^{-1}q, d=-q}.$$

For $\alpha \in \{2, 3, \dots\}$, let V_α be the α -dimensional irreducible $U_q(\mathfrak{sl}_2)$ module and $\varphi_\alpha: B_n \rightarrow \mathrm{GL}(V_\alpha^{\otimes n})$ be the quantum representation. Let $e: B_n \rightarrow \mathbb{Z}$ be the exponent sum map given by $e(\sigma_i^\pm) = \pm 1$. The representation φ_α can be recovered from a version of generic quantum representation as follows.

Proposition 2.4. *Let $\beta \in B_n$ and $\alpha \in \{2, 3, \dots\}$. Then*

$$e^{\frac{1}{4}\hbar(\alpha-1)^2 e(\beta)} \varphi_{\mathbb{L}}(\beta)|_{\lambda=\alpha-1} = \varphi_\alpha(\beta)$$

Proof. As we have seen, as $U_q(\mathfrak{sl}_2)$ -module we have an isomorphism $V_\alpha \cong V_{\mathbb{L}}|_{\lambda=\alpha-1}$. The formula follows from this observation and the definition $R = e^{-\frac{1}{4}\hbar\lambda^2} T\mathcal{R}$. \square

Next we observe when we specialize λ as integer, certain symmetry appears.

Lemma 2.5. $V_{n,m}|_{\lambda=\alpha-1} \cong V_{n,n\alpha-n-m}|_{\lambda=\alpha-1}$.

Proof. Let us put

$$R(v_i \otimes v_j) = \sum_{n=0}^{\infty} a_n v_{j+n} \otimes v_{i-n}, \quad R(v_{\lambda-j} \otimes v_{\lambda-i}) = \sum_{n=0}^{\infty} b_n v_{\lambda-i+n} \otimes v_{\lambda-j-n},$$

where $a_n, b_n \in \mathbb{L}|_{z=q^{\alpha-1}} \cong \mathbb{Z}[q^{\pm 1}]$. We show $a_n = b_n$ for all i, j . This shows an equivalence of R -operators hence proves the desired isomorphism.

Note that when the weight variable λ is specialized as a positive integer $\alpha - 1$, $v_k = 0$ whenever $k \geq \lambda$, $a_n = b_n = 0$ if $n > \min\{i, \lambda - j\}$. Hence we consider the case $n \leq \min\{i, \lambda - j\}$.

By (2.4), with putting $z = q^{\lambda-1}$, we have

$$\begin{aligned} b_n &= q^{\frac{\lambda}{2}(2\lambda-i-j)} q^{(\lambda-j-n)(\lambda-i+n)} q^{\frac{n(n-1)}{4}} \frac{[n+\lambda-i]_q!}{[n]_q! [\lambda-i]_q!} \prod_{k=0}^{n-1} (q^{\frac{1}{2}(j+k+1)} - q^{-\frac{1}{2}(j+k+1)}) \\ &= q^{-\frac{\lambda}{2}(i+j)} q^{(i-n)(j+n)} q^{\frac{n(n-1)}{4}} \frac{[n+\lambda-i]_q!}{[n]_q! [\lambda-i]_q!} \prod_{k=0}^{n-1} (q^{\frac{1}{2}(j+k+1)} - q^{-\frac{1}{2}(j+k+1)}) \end{aligned}$$

Since

$$\prod_{k=0}^{n-1} (q^{\frac{1}{2}(j+k+1)} - q^{-\frac{1}{2}(j+k+1)}) = \frac{[j+1]_q \cdots [j+n]_q}{(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^n} = \frac{[j+n]_q!}{[j]_q! (q^{\frac{1}{2}} - q^{-\frac{1}{2}})^n}$$

we conclude

$$\begin{aligned} b_n &= q^{-\frac{\lambda}{2}(i+j)} q^{(i-n)(j+n)} q^{\frac{n(n-1)}{4}} \frac{[n+\lambda-i]_q!}{[n]_q! [\lambda-i]_q!} \cdot \frac{[j+n]_q!}{[j]_q! (q^{\frac{1}{2}} - q^{-\frac{1}{2}})^n} \\ &= q^{-\frac{\lambda}{2}(i+j)} q^{(i-n)(j+n)} q^{\frac{n(n-1)}{4}} \frac{[j+n]_q!}{[n]_q! [j]_q!} \cdot \frac{[n+\lambda-i]_q!}{[\lambda-i]_q! (q^{\frac{1}{2}} - q^{-\frac{1}{2}})^n} \\ &= q^{-\frac{\lambda}{2}(i+j)} q^{(i-n)(j+n)} q^{\frac{n(n-1)}{4}} \frac{[j+n]_q!}{[n]_q! [j]_q!} \prod_{k=0}^{n-1} (q^{\frac{1}{2}(\lambda-i+k+1)} - q^{-\frac{1}{2}(\lambda-i+k+1)}) \\ &= a_n \end{aligned}$$

□

3. A TOPOLOGICAL FORMULA FOR THE LOOP EXPANSION OF THE COLORED JONES POLYNOMIALS

Now we are ready to prove our main result, a topological formula of the loop expansion of the colored Jones polynomials.

Theorem 3.1. *Let K be an oriented knot in S^3 represented as a closure of an n -braid β . Then the loop expansion of the colored Jones polynomial is given by*

$$CJ_K(\hbar, z) = \frac{z^{-\frac{1}{2}e(\beta)} q^{\frac{1}{2}e(\beta)}}{z^{\frac{1}{2}} - z^{-\frac{1}{2}}} \sum_{m=0}^{\infty} (z^{\frac{n}{2}} q^{\frac{1-n-2m}{2}} - z^{-\frac{n}{2}} q^{\frac{-1+n+2m}{2}}) \text{tr} L_{n,m}(\beta)|_{x=qz^{-1}, d=-q}.$$

Here we put $q = e^{\hbar}$.

Proof. The colored Jones polynomial $J_{K,\alpha}(q)$ is defined by

$$J_{K,\alpha}(q) = \frac{1}{[\alpha]_q} q^{-\frac{1}{4}(\alpha^2-1)e(\beta)} \text{tr}(q^{\frac{\hbar}{2}} \varphi_{\alpha}(\beta)).$$

By Proposition 2.4,

$$J_{K,\alpha}(q) = \frac{1}{[\alpha]_q} q^{-\frac{1}{4}(\alpha^2-1)e(\beta)} q^{\frac{1}{4}(\alpha-1)^2e(\beta)} \text{tr}(q^{\frac{\hbar}{2}} \varphi_{\mathbb{L}}(\beta)|_{\lambda=\alpha-1}).$$

The colored Jones function $CJ_K(z, \hbar)$ is obtained by taking the limit $\alpha \rightarrow \infty$ keeping $z = e^{\hbar\alpha}$ is constant, namely treating $\hbar\alpha$ as an independent variable:

$$CJ_K(\hbar, z) = \frac{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}{z^{\frac{1}{2}} - z^{-\frac{1}{2}}} z^{-\frac{1}{2}e(\beta)} q^{\frac{1}{2}e(\beta)} \lim_{\substack{\alpha \rightarrow \infty \\ \hbar\alpha: \text{constant}}} \text{tr}(q^{\frac{H}{2}} \varphi_{\mathbb{L}}(\beta)|_{\lambda=\alpha-1})$$

We compute the limit as follows (see Figure 1 for a diagrammatic summary of the computation.)

$$\begin{array}{ccccccc}
 q^{\frac{n\lambda}{2}} V_{n,0} & \subset & q^{\frac{n\lambda}{2}} \widehat{V}_{n,0} \cong_{\mathcal{Q}} & q^{\frac{n\lambda}{2}} \widehat{W}_{n,0} & & & \\
 \oplus & & \oplus & \oplus & & & \\
 q^{\frac{n\lambda-2}{2}} V_{n,1} & \subset & q^{\frac{n\lambda-2}{2}} \widehat{V}_{n,1} \cong_{\mathcal{Q}} & q^{\frac{n\lambda-2}{2}} \widehat{W}_{n,0} \oplus q^{\frac{n\lambda-2}{2}} \widehat{W}_{n,1} & & & \\
 \oplus & & \oplus & \oplus & & & \\
 \vdots & & \vdots & \vdots & & & \\
 \oplus & & \oplus & \oplus & & & \\
 q^{\frac{n\lambda-2k}{2}} V_{n,k} & \subset & q^{\frac{n\lambda-2k}{2}} \widehat{V}_{n,k} \cong_{\mathcal{Q}} & q^{\frac{n\lambda-2k}{2}} \widehat{W}_{n,0} \oplus q^{\frac{n\lambda-2k}{2}} \widehat{W}_{n,1} \oplus \cdots \oplus q^{\frac{n\lambda-2k}{2}} \widehat{W}_{n,k} & & & \\
 \oplus & & \oplus & \oplus & & & \\
 \vdots & & \vdots & \vdots & & & \\
 \oplus & & \oplus & \oplus & & & \\
 q^{\frac{-n\lambda+2k}{2}} V_{n,n\lambda-k} \subset^* & q^{\frac{-n\lambda+2k}{2}} \widehat{V}_{n,k} \cong_{\mathcal{Q}} & q^{\frac{-n\lambda+2k}{2}} \widehat{W}_{n,0} \oplus q^{\frac{-n\lambda+2k}{2}} \widehat{W}_{n,1} \oplus \cdots \oplus q^{\frac{-n\lambda+2k}{2}} \widehat{W}_{n,k} & & & & \\
 \oplus & & \oplus & & & & \\
 \vdots & & \vdots & & & & \\
 \oplus & & \oplus & & & & \\
 q^{\frac{n\lambda-1}{2}} V_{n,n\lambda-1} \subset^* & q^{\frac{-n\lambda+2}{2}} \widehat{V}_{n,1} \cong_{\mathcal{Q}} & q^{\frac{-n\lambda+2}{2}} \widehat{W}_{n,0} \oplus q^{\frac{-n\lambda+2}{2}} \widehat{W}_{n,1} & & & & \\
 \oplus & & \oplus & & & & \\
 q^{\frac{-n\lambda}{2}} V_{n,n\lambda} \subset^* & q^{\frac{-n\lambda}{2}} \widehat{V}_{n,0} \cong_{\mathcal{Q}} & q^{\frac{-n\lambda}{2}} \widehat{W}_{n,0} & & & & \\
 \uparrow & & \parallel & & \parallel & & \parallel \\
 \boxed{\text{As } \alpha \rightarrow \infty \cong} & & [n\lambda+1]_q \widehat{W}_{n,0} & & [n\lambda-1]_q \widehat{W}_{n,1} & & [n\lambda+1-2k]_q \widehat{W}_{n,k}
 \end{array}$$

FIGURE 1. This diagram explains how to compute the desired limit. In the diagram, every representations are understood as taking specialization $\lambda = \alpha - 1$. The notation \subset^* means that we regard $V_{n,n\lambda-i}$ as sub module of $\widehat{V}_{n,i}$, by using isomorphism $V_{n,n\lambda-i} \cong V_{n,i}$ in Lemma 2.5.

Since $V_{n,i}|_{\lambda=\alpha-1} = 0$ if $i > n\lambda = n(\alpha - 1)$, by (2.7), as a $\mathbb{L}B_n$ -module $V_{\mathbb{L}}^{\otimes n} \cong \bigoplus_{i=0}^{n\lambda} V_{n,i}$. Moreover, $q^{\frac{H}{2}}$ acts on $V_{n,i}$ as a scalar multiple by $q^{\frac{n\lambda-2i}{2}}$, so

$$\text{tr}(q^{\frac{H}{2}} \varphi_{\mathbb{L}}(\beta)|_{\lambda=\alpha-1}) = \sum_{i=0}^{n\lambda} q^{\frac{n\lambda-2i}{2}} \text{tr}(\varphi_{n,i}^V(\beta)|_{\lambda=\alpha-1}).$$

By Lemma 2.5, we identify the braid group representation $V_{n,n\lambda-k}$ as $V_{n,k}$ for $k \leq \frac{n\lambda}{2}$, and then regard each $V_{n,i}$ as a sub $\mathbb{L}B_n$ -module of $\widehat{V}_{n,i}$. Recall that $\text{tr}(\varphi_{n,i}^V(\beta)|_{\lambda=\alpha-1})$ is equal to $\text{tr}(\widehat{\varphi_{n,i}^V}(\beta)|_{\lambda=\alpha-1})$ when α is treated as independent variable. By (2.6), over quotient field, $\widehat{V}_{n,i}$ splits as $\bigoplus_{m=0}^i \widehat{W}_{n,m}$, hence

$$\text{tr}(\widehat{\varphi_{n,i}^V}(\beta)|_{\lambda=\alpha-1}) = \sum_{m=0}^{\min\{m, n\lambda-m\}} \text{tr}(\widehat{\varphi_{n,m}^W}(\beta)|_{\lambda=\alpha-1})$$

This shows

$$\begin{aligned}
\sum_{i=0}^{\alpha-1} q^{\frac{n\lambda-2i}{2}} \operatorname{tr}(\widehat{\varphi_{n,i}^V}(\beta)|_{\lambda=\alpha-1}) &= \sum_{i=0}^{\alpha-1} q^{\frac{n\lambda-2i}{2}} \sum_{m=0}^{\min\{m, n\lambda-m\}} \operatorname{tr}(\widehat{\varphi_{n,i}^W}(\beta)|_{\lambda=\alpha-1}) \\
&= \sum_{m=0}^{\frac{n\lambda}{2}} \sum_{i=m}^{n\lambda-m} q^{\frac{n\lambda-2i}{2}} \operatorname{tr}(\widehat{\varphi_{n,m}^W}(\beta)|_{\lambda=\alpha-1}) \\
&= \sum_{m=0}^{\frac{n\lambda}{2}} [n\lambda + 1 - 2m]_q \operatorname{tr}(\widehat{\varphi_{n,m}^W}(\beta)|_{\lambda=\alpha-1})
\end{aligned}$$

By Theorem 2.3, when we treat α as independent variable,

$$\operatorname{tr}(\widehat{\varphi_{n,i}^W}(\beta)) = \operatorname{tr}(L_{n,m}(\beta)|_{x=z^{-1}q, d=-q})$$

hence

$$\begin{aligned}
\lim_{\substack{\alpha \rightarrow \infty \\ h\alpha: \text{constant}}} \operatorname{tr}(q^{\frac{H}{2}} \circ \varphi_{\mathbb{L}}(\beta)|_{\lambda=\alpha-1}) &= \lim_{\substack{\alpha \rightarrow \infty \\ h\alpha: \text{constant}}} \sum_{m=0}^{\min\{m, n\lambda-m\}} [n\lambda + 1 - 2m]_q \operatorname{tr}(L_{n,m}(\beta)|_{x=z^{-1}q, d=-q}) \\
&= \sum_{m=0}^{\infty} \frac{z^{\frac{n}{2}} q^{\frac{1}{2}(1-n-2m)} - z^{-\frac{n}{2}} q^{-\frac{1}{2}(1-n-2m)}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \operatorname{tr}(L_{n,m}(\beta)|_{x=z^{-1}q, d=-q}).
\end{aligned}$$

Therefore we conclude

$$CJ_K(\hbar, z) = \frac{z^{-\frac{1}{2}e(\beta)} q^{\frac{1}{2}e(\beta)}}{z^{\frac{1}{2}} - z^{-\frac{1}{2}}} \sum_{m=0}^{\infty} (z^{\frac{n}{2}} q^{\frac{1}{2}(1-n-2m)} - z^{-\frac{n}{2}} q^{-\frac{1}{2}(1-n-2m)}) \operatorname{tr}(L_{n,m}(\beta)|_{x=z^{-1}q, d=-q}).$$

□

As we have mentioned, Theorem 3.1 provides an alternative, direct method to compute the loop expansion of the colored Jones polynomial although actual computation may be quite hard, since one should know $L_{n,m}(\beta)$ for all m . Here we give sample calculations.

Example 3.2 (Unknot). Let us consider the unknot K represented as a closure of 2-braid σ_1 . The trace of Lawrence's representation is given by $\operatorname{tr} L_{2,m}(\sigma_1) = (-x)^m (-d)^{\binom{m}{2}}$ so

$$\begin{aligned}
CJ_{\text{Unknot}}(z, \hbar) &= \frac{z^{-\frac{1}{2}} q^{\frac{1}{2}}}{z^{\frac{1}{2}} - z^{-\frac{1}{2}}} \sum_{m=0}^{\infty} (z q^{\frac{-1-2m}{2}} - z^{-1} q^{\frac{1+2m}{2}}) (-z)^{-m} q^{\binom{m+1}{2}} \\
&= \frac{z^{-\frac{1}{2}} q^{\frac{1}{2}}}{z^{\frac{1}{2}} - z^{-\frac{1}{2}}} (z q^{-\frac{1}{2}} - q^{-\frac{1}{2}}) \\
&= 1.
\end{aligned}$$

Example 3.3 $((2, p)$ -torus knot). More generally, let us consider $(2, p)$ -torus knot $T(2, p)$ represented as a closure of 2-braid σ_1^p . The trace of Lawrence's representation is given by $\operatorname{tr} L_{2,m}(\sigma_1^p) = (-x^p)^m (-d^p)^{\binom{m}{2}}$ so

$$(3.1) \quad CJ_{T(2,p)}(z, \hbar) = \frac{z^{-\frac{1}{2}p} q^{\frac{1}{2}p}}{z^{\frac{1}{2}} - z^{-\frac{1}{2}}} \sum_{m=0}^{\infty} (z q^{\frac{-1-2m}{2}} - z^{-1} q^{\frac{1+2m}{2}}) (-z^{-p})^m q^{\binom{m+1}{2}p}$$

To compute the 1-loop part, let us put $\hbar = 0$. Then

$$\begin{aligned}
V_{T(2,p)}^{(0)}(z) &= \frac{z^{-\frac{1}{2}p}}{z^{\frac{1}{2}} - z^{-\frac{1}{2}}} \sum_{m=0}^{\infty} (z - z^{-1}) (-z^{-p})^m = (z^{\frac{1}{2}} + z^{-\frac{1}{2}}) z^{-\frac{p}{2}} \sum_{m=0}^{\infty} (-z^{-p})^m \\
&= (z^{\frac{1}{2}} + z^{-\frac{1}{2}}) z^{-\frac{p}{2}} \frac{1}{1 + z^{-p}} \\
&= \frac{z^{\frac{1}{2}} + z^{-\frac{1}{2}}}{z^{\frac{1}{2}p} + z^{-\frac{1}{2}p}}
\end{aligned}$$

which is equal the inverse of the Alexander-Conway polynomial of $T(2, p)$.

Next let us compute the 2-loop part. By putting $q = e^{\hbar}$ and looking at the coefficient of \hbar in (3.1), we have

$$V_{T(2,p)}^{(1)}(z) = \frac{z^{-\frac{1}{2}p}}{z^{\frac{1}{2}} - z^{-\frac{1}{2}}} \left(\frac{z}{2} \sum_{m=0}^{\infty} [pm^2 + (p-2)m + (p-1)](-z^{-pm}) - \frac{z^{-1}}{2} \sum_{m=0}^{\infty} [pm^2 + (p+2)m + (p+1)](-z^{-pm}) \right)$$

In the ring of formal power series $\mathbb{C}[[z, z^{\pm 1}]]$, we have

$$\sum_{m=0}^{\infty} z^m = \frac{1}{1-z}, \quad \sum_{m=0}^{\infty} mz^m = \frac{z}{(1-z)^2}, \quad \sum_{m=0}^{\infty} m^2 z^m = \frac{z+z^2}{(1-z)^3}.$$

Hence

$$V_{T(2,p)}^{(1)}(z) = \frac{1}{(z^{\frac{1}{2}} - z^{-\frac{1}{2}})(z^{\frac{1}{2}p} + z^{-\frac{1}{2}p})^3} ((p-1)z^{p+1} - (p+1)z^{p-1} + (p+1)z^{-p+1} - (p-1)z^{-p-1}).$$

For example,

$$V_{T(2,3)}^{(1)}(z) = \frac{(z^4 - 2z^2 + 2 - 2z^{-2} + z^{-4})}{\Delta_{T(2,3)}(z)^3} = \frac{(z^4 - 2z^2 + 2 - 2z^{-2} + z^{-4})}{(z - 1 + z^{-1})^3}.$$

Now it is a direct consequence that the 1-loop part is the inverse of the Alexander-Conway polynomial.

Corollary 3.4 (Melvin-Morton-Ronzansky conjecture).

$$V^{(0)}(z) = \frac{1}{\Delta_K(z)}$$

Proof. The 1-loop part $V^{(0)}(z)$ is obtained by putting $\hbar = 0$ in the formula of Theorem 3.1. Since $d = -q = -e^{\hbar}$, in a homological representation, putting $\hbar = 0$ corresponds to putting $d = -1$. As we have pointed out in Proposition 2.2,

$$L_{n,m}(\beta)|_{d=-1} = \text{Sym}^m L_{n,1}(\beta).$$

Therefore, by Theorem 3.1, the 1-loop part $V^{(0)}(z)$ is written as

$$\begin{aligned} V^{(0)}(z) &= z^{-\frac{1}{2}e(\beta)} \frac{z^{\frac{n}{2}} - z^{-\frac{n}{2}}}{z^{\frac{1}{2}} - z^{-\frac{1}{2}}} \sum_{i=0}^{\infty} \text{tr} L_{n,i}(\beta)|_{x=z^{-1}, d=-1} \\ &= z^{-\frac{1}{2}e(\beta)} \frac{z^{\frac{n}{2}} - z^{-\frac{n}{2}}}{z^{\frac{1}{2}} - z^{-\frac{1}{2}}} \sum_{i=0}^{\infty} \text{tr}(\text{Sym}^i L_{n,1})(\beta)|_{x=z^{-1}}. \end{aligned}$$

MacMahon Master Theorem says that

$$\sum_{i=0}^{\infty} \text{tr}(\text{Sym}^i L_{n,1})(\beta) = \det(I - L_{n,1}(\beta))^{-1}$$

hence

$$V^{(0)}(z) = z^{-\frac{1}{2}e(\beta)} \frac{z^{\frac{n}{2}} - z^{-\frac{n}{2}}}{z^{\frac{1}{2}} - z^{-\frac{1}{2}}} \frac{1}{\det(I - L_{n,1}(\beta)|_{x=z^{-1}})} = \frac{1}{\Delta_K(z^{-1})} = \frac{1}{\Delta_K(z)}.$$

□

4. ENTROPY AND COLORED JONES POLYNOMIALS

In this section we give an application of topological interpretation of quantum representations.

4.1. Entropy estimates from configuration space. For a homeomorphism of a compact topological space or a metric space $f : X \rightarrow X$, there is a fundamental numerical invariant $h(f) \in \mathbb{R}$ of topological dynamics called the (*topological*) *entropy*.

Let $\overline{C}_m(X)$ and $C_m(X)$ be the *ordered* and *unordered configuration space* of m -points of X ,

$$\overline{C}_m(X) = \{(x_1 \dots, x_m) \in X^m \mid x_i \neq x_j\}, \quad C_m(X) = \overline{C}_m(X)/S_m$$

where S_m is the symmetric group that acts as permutations of the coordinates. Then f induces the continuous maps $\overline{C}_m(f) : \overline{C}_m(X) \rightarrow \overline{C}_m(X)$ and $C_m(f) : C_m(X) \rightarrow C_m(X)$, respectively.

Note that $\overline{C}_m(X) \subset X^m$ is invariant under $f^{\times m} : X^m \rightarrow X^m$ so

$$h(\overline{C}_m(f)) \leq h(f^{\times m}) = mh(f).$$

The unordered configuration space $\overline{C}_m(X)$ is a finite cover of $C_m(X)$ so $h(C_m(f)) = h(\overline{C}_m(f))$.

Now for $A \in \mathrm{GL}(n; \mathbb{C})$ let $\rho(A)$ be its spectral radius of A , the maximum of the absolute value of the eigenvalues of A . It is known that if $C_m(f)$ is nice enough (see [Fr], for sufficient conditions for inequality (4.1) to hold), then spectral radius of the induced action on homology provides a lower bound of the entropy

$$(4.1) \quad \log \rho(C_m(f)_* : H_*(C_m(f), \mathbb{Z}) \rightarrow H_*(C_m(f), \mathbb{Z})) \leq h(C_m(f)).$$

Hence if f is good enough, by using configuration spaces we have an estimate of entropy

$$(4.2) \quad \log \rho(C_m(f)_*) \leq mh(f).$$

The above considerations nicely fit for the braid groups. Let us regard the braid group B_n as the mapping class group of n -punctured disc D_n . The entropy of braid $\beta \in B_n$ is defined by the infimum of entropy of homeomorphisms representing β ,

$$h(\beta) = \inf \{h(f) \mid f : D_n \rightarrow D_n, [f] = \beta \in \mathrm{MCG}(D_n) = B_n\}.$$

By Nielsen-Thurston classification [FLP, Th], there is a representative homeomorphism f_β that attains the infimum so $h(\beta) = h(f_\beta)$. In particular, if β is pseudo-Anosov, then a pseudo-Anosov representative attains the infimum. By abuse of notation, we will use the same symbol β to mean its representative homeomorphism f_β that attains the infimum of the entropy.

As Koberda shows in [Kob], the inequality (4.1) holds in the case X is surface. This implies that Lawrence's representation gives an estimate of entropy.

Theorem 4.1. *For an n -braid β ,*

$$\sup_{|x|=1, |d|=1} \log \rho(L_{n,m}(\beta)) \leq mh(\beta)$$

Proof. Let \tilde{C} be a finite covering of the unordered configuration space $C_{n,m} = C_m(D_n)$. If the action of β on $C_{n,m}$ lifts, then by (4.2), $\log \rho(\beta_{\tilde{C}*}) \leq mh(\beta)$ holds, where $\beta_{\tilde{C}} : \tilde{C} \rightarrow \tilde{C}$ denotes the lift of a homeomorphism β .

For non-negative integers A, B , Let $\tilde{C} = \tilde{C}_{A,B}$ be a finite abelian covering of $C_{n,m}$ that corresponds to the kernel of $\alpha_{A,B} : \pi_1(C_{n,m}) \rightarrow \mathbb{Z}/A\mathbb{Z} \oplus \mathbb{Z}/B\mathbb{Z}$, where $\alpha_{A,B}$ is given by the compositions

$$\pi_1(C_{n,m}) \xrightarrow{\alpha} \mathbb{Z} \oplus \mathbb{Z} \cong \langle x \rangle \oplus \langle d \rangle \longrightarrow (\langle x \rangle / x^A) \oplus (\langle d \rangle / d^B) = \mathbb{Z}/A\mathbb{Z} \oplus \mathbb{Z}/B\mathbb{Z}.$$

The standard topological argument, using the eigenspace decompositions for the deck translations (See [BB] for the case \mathbb{Z} -covering, the case of the reduced Burau representation $L_{n,1}$. The same argument applies to the case \mathbb{Z}^2 -covering) shows that for $a = 1, \dots, A-1$ and $b = 1, \dots, B-1$,

$$\rho(L_{n,m}(\beta))|_{x=e^{\frac{2\pi a\sqrt{-1}}{A}}, d=e^{\frac{2\pi b\sqrt{-1}}{B}}} \leq \rho(\tilde{\beta}_{\tilde{C}_{A,B}}).$$

The sets $\{(e^{\frac{2\pi a\sqrt{-1}}{A}}, e^{\frac{2\pi b\sqrt{-1}}{B}}) \in |A, a, B, b \in \mathbb{Z}\}$ is dense in $S^1 \times S^1 = \{(x, d) \in \mathbb{C}^2 \mid |x| = |d| = 1\}$, hence we get desired inequality. \square

Since generically one can identify the quantum representation with Lawrence's representation, quantum representations also provide estimates of entropy.

Theorem 4.2. *Let β be an n -braid.*

- (1) $\sup_{|q|=1, |z|=1} \log \rho(\widehat{\varphi_{n,m}^W}(\beta)) \leq mh(\beta).$
- (2) $\sup_{|q|=1, |z|=1} \log \rho(\widehat{\varphi_{n,m}^V}(\beta)) \leq mh(\beta).$
- (3) $\sup_{|q|=1} \log \rho(\varphi_\alpha(\beta)) \leq \frac{n\alpha-n}{2}h(\beta).$

Proof. The assertions (1) and (2) follows from Theorem 4.1 and Theorem 2.3. To see (3), recall that as a $\mathbb{L}B_n|_{z=q^{\alpha-1}} = \mathbb{C}[q^{\pm 1}]B_n$ -module, we have

$$V_\alpha^{\otimes n} \cong V_{\mathbb{L}}|_{z=q^{\alpha-1}} \subset \bigoplus_{m=0}^{n(\alpha-1)} V_{n,m}|_{\lambda=\alpha-1}.$$

Moreover, by Lemma 2.5 $V_{n,m}|_{\lambda=\alpha-1} \cong V_{n,n(\alpha-1)-m}|_{\lambda=\alpha-1}$. Therefore,

$$\sup_{|q|=1} \rho(\varphi_\alpha(\beta)) \leq \max_{1 \leq m \leq \frac{n(\alpha-1)}{2}} \sup_{|q|=1} (\varphi_{n,m}^V(\beta)|_{z=q^{\alpha-1}}) \leq \max_{1 \leq m \leq \frac{n(\alpha-1)}{2}} \sup_{|q|=1} \rho(\widehat{\varphi_{n,m}^V}(\beta)|_{z=q^{\alpha-1}}).$$

By (1), we conclude

$$\sup_{|q|=1} \log \rho(\varphi_\alpha(\beta)) \leq \frac{n(\alpha-1)}{2}h(\beta).$$

□

4.2. Quantum \mathfrak{sl}_2 invariants and entropy. An estimates in Theorem 4.2 suggests a new relationship between quantum invariants and entropy of braids.

For $\alpha \in \{2, 3, \dots\}$, let $Q_K^{\mathfrak{sl}_2; V_\alpha}(q) = \text{tr}(q^{\frac{H}{2}} \varphi_\alpha(\beta)) = [\alpha]_q J_{\alpha, K}(q)$ be the quantum $(\mathfrak{sl}_2, V_\alpha)$ -invariant of the knot K , another common normalization of the colored Jones polynomials used to define quantum invariants of 3-manifolds.

Theorem 4.3. *Let K be a knot represented as the closure of an n -braid β , and $\alpha \in \{2, 3, \dots\}$. Then*

$$\sup_{|q|=1} \log |Q_K^{\mathfrak{sl}_2; V_\alpha}(q)| \leq n \log \alpha + \log \rho(\varphi_\alpha(\beta)) \leq n \log \alpha + \frac{n(\alpha-1)}{2}h(\beta).$$

Proof. By definition of the spectral radius,

$$|Q_K^{\mathfrak{sl}_2; V_\alpha}(q)| = |\text{tr}(q^{\frac{H}{2}} \varphi_\alpha(\beta))| \leq \alpha^n \rho(q^{\frac{H}{2}} \varphi_\alpha(\beta)) \leq \alpha^n \rho(q^{\frac{H}{2}}) \rho(\varphi_\alpha(\beta)).$$

Here the last inequality follows from the fact that $q^{\frac{H}{2}}$ and $\varphi_\alpha(\beta)$ commutes. When $|q| = 1$, $\rho(q^{\frac{H}{2}}) = 1$ hence by Theorem 4.2 (3), we conclude

$$\sup_{|q|=1} |\text{tr}(q^{\frac{H}{2}} \varphi_\alpha(\beta))| \leq \sup_{|q|=1} \alpha^n \rho(\varphi_\alpha(\beta)) \leq \alpha^n e^{\frac{n(\alpha-1)}{2}h(\beta)}.$$

□

By an analogy of the famous volume conjecture [Ka, MuMu], It is interesting to look at the asymptotic behavior of $|Q_K^{\mathfrak{sl}_2; V_\alpha}(q)|$. By Theorem 4.3, we have

$$\frac{\sup_{|q|=1} \log |Q_K^{\mathfrak{sl}_2; V_\alpha}(q)|}{\alpha} \leq n \frac{\log \alpha}{\alpha} + \frac{\sup_{|q|=1} \log \rho(\varphi_\alpha(\beta))}{\alpha} \leq n \frac{\log \alpha}{\alpha} + \frac{n(\alpha-1)}{2\alpha}h(\beta)$$

This shows

$$(4.3) \quad \limsup_{\alpha \rightarrow \infty} \frac{\sup_{|q|=1} \log |Q_K^{\mathfrak{sl}_2; V_\alpha}(q)|}{\alpha} \leq \limsup_{\alpha \rightarrow \infty} \frac{\sup_{|q|=1} \log \rho(\varphi_\alpha(\beta))}{\alpha} \leq \frac{n}{2}h(\beta).$$

It is interesting to ask the convergence of the limits and when the inequalities (4.3) yield the equalities. In particular, the second inequality is related to the question when the estimation of entropy from quantum representation is asymptotically sharp.

A. APPENDIX: MULTIFORKS FOR LAWRENCE'S REPRESENTATION $L_{n,m}$

In this appendix, we present multiforks in Lawrence's representation $L_{n,m}$ and explicit matrices of $L_{n,m}(\sigma_i)$. For the basics of geometric treatments of Lawrence's representation, see [I, Section 2].

First we review the definition of multiforks and how multifork represent a homology class in $H_m^{lf}(\widetilde{C_{n,m}}; \mathbb{Z})$. Let Y be the Y -shaped graph with four vertices c, r, v_1, v_2 and oriented edges as shown in Figure 2(1). A *fork* F based on $d \in \partial D_n$ is an embedded image of Y into $D^2 = \{z \in \mathbb{C} \mid |z| \leq n+1\}$ such that:

- All points of $Y \setminus \{r, v_1, v_2\}$ are mapped to the interior of D_n .
- The vertex r is mapped to d_i .
- The other two external vertices v_1 and v_2 are mapped to the puncture points.

The image of the edge $[r, c]$ and the image of $[v_1, v_2] = [v_1, c] \cup [c, v_2]$ regarded as a single oriented arc, are denoted by $H(F)$ and $T(F)$. We call $H(F)$ and $T(F)$ the *handle* and the *tine* of the fork F , respectively.

A *multifork* of dimension m is an ordered tuples of m forks $\mathbb{F} = (F_1, \dots, F_m)$ such that

- F_i is a fork based on d_i .
- $T(F_i) \cap T(F_j) \cap D_n = \emptyset$ ($i \neq j$).
- $H(F_i) \cap H(F_j) = \emptyset$ ($i \neq j$).

Figure 2 (2) illustrates an example of a multifork of dimension 3. We often use to represent multiforks consisting of k parallel forks by drawing single fork labelled by k , as shown in Figure 2 (3).

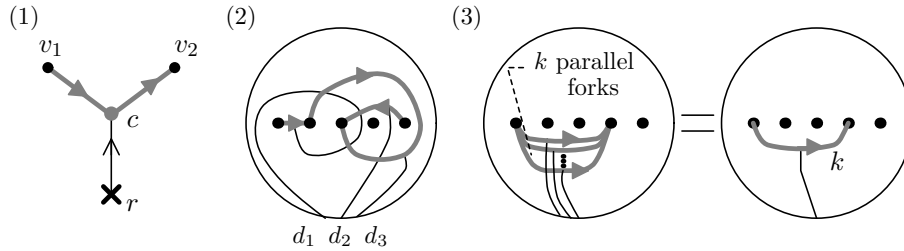


FIGURE 2. Multiforks: to distinguish tines and handle, we often write tine of forks by a bold gray line.

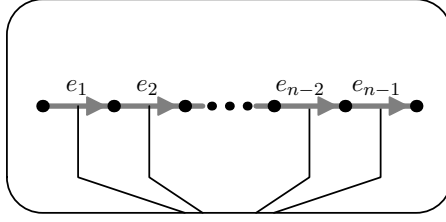
For a multifork \mathbb{F} , we regard the handle $H(F_i)$ of the fork F_i as a path $\gamma_i: [0, 1] \rightarrow D_n$ in D_n by taking an appropriate parametrization. Then the handles of \mathbb{F} defines a path $H(\mathbb{F}) = \{\gamma_1, \dots, \gamma_m\}: [0, 1] \rightarrow C_{n,m}$ in $C_{n,m}$. Take a lift of $H(\mathbb{F})$, $\widetilde{H}(\mathbb{F}): [0, 1] \rightarrow \widetilde{C_{n,m}}$ so that $\widetilde{H}(\mathbb{F})(0) = \widetilde{\mathbf{d}}$.

Let $\Sigma(\mathbb{F}) = \{\{z_1, \dots, z_m\} \in C_{n,m} \mid z_i \in T(F_i)\}$, and $\widetilde{\Sigma}(\mathbb{F})$ be the m -dimensional submanifold of $\widetilde{C_{n,m}}$ which is the connected component of $\pi^{-1}(\Sigma(\mathbb{F}))$ containing $\widetilde{H}(\mathbb{F})(1)$. The submanifold $\widetilde{\Sigma}(\mathbb{F})$ defines an element of $H_m^{lf}(\widetilde{C_{n,m}}; \mathbb{Z})$. By abuse of notation, we will use \mathbb{F} to represent both multifork and the homology class $[\widetilde{\Sigma}(\mathbb{F})] \in H_m^{lf}(\widetilde{C_{n,m}}; \mathbb{Z})$.

Here the orientation of $\widetilde{\Sigma}(\mathbb{F})$ is defined so that a canonical homeomorphism $T(F_1) \times \dots \times T(F_m) \rightarrow \Sigma(\mathbb{F})$ is orientation preserving. Thus, for a fork $\mathbb{F}_\tau = (F_{\tau(1)}, \dots, F_{\tau(m)})$ obtained by permuting its coordinate by a permutation $\tau \in S_m$, we have $\mathbb{F}_\tau = \text{sgn}(\tau)\mathbb{F} \in H_m^{lf}(\widetilde{C_{n,m}}; \mathbb{Z})$.

For $\mathbf{e} = (e_1, \dots, e_{n-1}) \in E_{n,m}$, we assign a multifork $\mathbb{F}_\mathbf{e} = \{F_1, \dots, F_m\}$ in Figure 3 and call \mathbb{F} a *standard multifork*.

The set of standard multiforks spans a $\mathbb{Z}[x^{\pm 1}, d^{\pm 1}]$ -submodule $\mathcal{H}_{n,m}$ of $H_m^{lf}(\widetilde{C_{n,m}}; \mathbb{Z})$, which is free of dimension $\binom{n+m-2}{2}$ and is invariant under the B_n -action. This defines a (geometric) Lawrence's representation $L_{n,m}: B_n \rightarrow \text{GL}(\binom{n+m-2}{2}; \mathbb{Z}[x^{\pm 1}, d^{\pm 1}])$.


 FIGURE 3. Standard multifork $\mathbb{F}_{\mathbf{e}}$ for $\mathbf{e} = (e_1, \dots, e_{n-1})$

From the definition of the submanifold $\tilde{\Sigma}(\mathbb{F})$, we graphically express several relations among homology classes represented by multifork, which allows us to express a given multifork as a sum of standard multiforks (see [Kra, Big2] for the case $m = 2$).

$$\begin{aligned}
 \text{(F1)} \quad & \text{A horizontal line with a dot and a vertical line segment extending downwards, with a curved arrow labeled } k \text{ pointing to the right, is equal to } (-1)^k \text{ times a horizontal line with a dot and a vertical line segment extending downwards, with a curved arrow labeled } k \text{ pointing to the left.} \\
 \text{(F2)} \quad & \text{A horizontal line with a dot and a vertical line segment extending downwards, with a loop labeled } k \text{ on the left, is equal to } (-d)^{\binom{k}{2}} x^k \text{ times a horizontal line with a dot and a vertical line segment extending downwards.} \\
 \text{(F3)} \quad & \text{A horizontal line with a dot and a vertical line segment extending downwards, with a horizontal line segment labeled } k \text{ above it, is equal to } x^k \text{ times a horizontal line with a dot and a vertical line segment extending downwards, with a horizontal line segment labeled } k \text{ above it.} \\
 \text{(F4)} \quad & \text{A horizontal line with a dot and a vertical line segment extending downwards, with a horizontal line segment labeled } k \text{ above it, is equal to a sum over } i \text{ from } 0 \text{ to } k \text{ of } \binom{k}{i} \text{ times a horizontal line with a dot and a vertical line segment extending downwards, with a horizontal line segment labeled } k-i \text{ above it and a dot labeled } i \text{ on the right.} \\
 \text{(F4)} \quad & \text{A horizontal line with a dot and a vertical line segment extending downwards, with a horizontal line segment labeled } k \text{ above it, is equal to a sum over } i \text{ from } 0 \text{ to } k \text{ of } \binom{k}{i} \text{ times a horizontal line with a dot and a vertical line segment extending downwards, with a horizontal line segment labeled } k-i \text{ above it and a dot labeled } i \text{ on the left.}
 \end{aligned}$$

FIGURE 4. Geometric rewriting formula for multiforks

In particular, these formulae leads to a formula of an explicit matrix representative of $L_{n,m}(\sigma_i)$:

$$\begin{aligned}
 \text{(A.1)} \quad & L_{n,m}(\sigma_1)(\mathbb{F}_{e_1, \dots, e_{n-1}}) = \sum_{l=0}^{e_2} (-1)^{e_1} (-d)^{\binom{e_1}{2}} x^{e_1} \binom{e_2}{l} \mathbb{F}_{e_1+e_2-l, l, \dots} \\
 & L_{n,m}(\sigma_i)(\mathbb{F}_{e_1, \dots, e_{n-1}}) = \sum_{k=0}^{e_{i-1}} \sum_{l=0}^{e_{i+1}} (-1)^{e_i} (-d)^{\binom{e_i+k}{2}} x^{e_i+k} \binom{e_{i-1}}{k} \binom{e_{i+1}}{l} \mathbb{F}_{\dots, e_{i-1}-k, e_i+k+e_{i+1}-l, \dots} \\
 & \quad (i = 2, \dots, n-2) \\
 & L_{n,m}(\sigma_{n-1})(\mathbb{F}_{e_1, \dots, e_{n-1}}) = \sum_{k=0}^{e_{n-2}} (-1)^{e_{n-1}} (-d)^{\binom{e_{n-1}+k}{2}} x^{e_{n-1}+k} \binom{e_{n-2}}{k} \mathbb{F}_{\dots, e_{n-2}-k, e_{n-1}+k}
 \end{aligned}$$

As we already mentioned, Proposition 2.2 follows from formula (A.1).

However, a multifork expression gives a direct way to see Proposition 2.2: First note that by orientation convention of $\tilde{\Sigma}(\mathbb{F})$, when $d = -1$, the homology class represented by a multifork $\mathbb{F} = (F_1, \dots, F_m)$ is independent of a choice of indices of forks, namely, for a fork $\mathbb{F}_\tau = (F_{\tau(1)}, \dots, F_{\tau(m)})$ obtained by permuting its coordinate by a permutation $\tau \in S_m$, $\mathbb{F}_\tau = \mathbb{F}$. Therefore, the correspondence between multifork $\mathbb{F} = (F_1, \dots, F_m)$ that represents an element of $\mathcal{H}_{n,m}$ and a family of m

forks $\{F_1, \dots, F_k\}$ that represents an element of $\text{Sym}^m \mathcal{H}_{n,1}$ gives rise to the desired isomorphism $\mathcal{H}_{n,m} \rightarrow \text{Sym}^m \mathcal{H}_{n,1}$.

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